refrigerator of s.s. 'Ascanius' in Liverpool. These experiments could not have been undertaken had it not been for the help kindly given by Captain Chrimes, Chief Engineer Douglas, and Refrigerating Engineer Latham, of s.s. 'Ascanius,' and by the Superintendent of the P. and O. Company in Sydney, the Chief Officers, the Purser, and Mr. Charlwood, of R.M.S. 'Morea.' The writer is also indebted to Mr. C. C. Pearce for assistance with the diagrams.

It was through the hospitality of Prof. Kerr Grant, of the Physical Laboratory of the Adelaide University, and through the skill of Mr. Rogers, of the workshop there, that it was possible so to modify the apparatus that a test was possible on the homeward voyage.

On the Determination of Gravity at Sea (Note on Dr. Duffield's Paper).

By Arthur Schuster, Sec. R.S.

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1. In a paper recently communicated to this Society, Dr. Duffield has described a promising method of determining gravity at sea, but a somewhat fuller discussion on the theoretical side is required before an opinion of its capabilities and limitations can be formed. For this purpose it is more especially necessary to investigate the effects of the forced vibrations which may be imposed on the mercury in the apparatus by the vertical acceleration of the ship.

2. The problem of determining the period of free oscillation in a barometer tube is very simple, but in the present case three tubes are connected with each other, and at least two of the tubes are not of uniform section, but contain a portion with a narrow bore introduced to damp the oscillations.

Starting with a single tube (fig. 1), let a glass vessel closed at the top and dipping into a trough of mercury at its lower end consist of a series of tubes having different diameters. The upper part of the vessel is supposed to contain air at a reduced pressure, and the mercury is disturbed from its equilibrium position at H. In each section of the tube having a radius r_0, the velocity of the flow at a point which is at a distance r from the central axis is $v = v_0 (r_0^2 - r^2)/r_0^2$, where v_0 is the velocity along the axis. From this it follows that the mean velocity and the mean square of the velocity

over the cross-section are respectively $\frac{1}{2}v_0$ and $\frac{1}{3}v_0^2$. The potential and kinetic energies are most suitably expressed by the quantity Q of mercury which stands above the equilibrium position H. If s be the area of the cross-section, dQ/dt (the rate at which the volume Q is transmitted through the tube) is equal to $\frac{1}{2}sv_0$. Hence the mean square of the velocity of matter flowing across the section is $\frac{1}{3}v_0^2 = \frac{4}{3}s^{-2}(dQ/dt)^2$. If there be a series of tubes joined together of length l', l'', etc., and cross-sections s', s'', etc., the total kinetic energy in the vessel is $\frac{2}{3}\rho(dQ/dt)^2\Sigma l/s$.

If one of the cross-sections be small, compared with the rest, it will contribute the largest share to the kinetic energy, and, if sufficiently small,

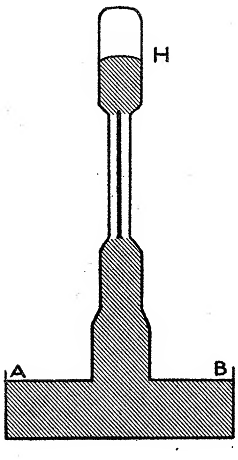

FIG. 1.

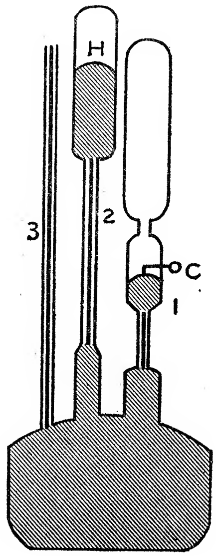

FIG. 2.

it alone need be taken into consideration; but we may retain the more general expression for the present.

As regards the potential energy, we take h_0 to be the vertical height of the column of mercury in its position of equilibrium. If a quantity of mercury Q has entered the tube, the upper level of the mercury has been raised through a distance Q/a, a being the cross-section at H. The work done against gravity is therefore $g\rho Q(h_0 + \frac{1}{2}Q/a)$. If V be the original volume of the air in the upper part of the vessel, and P its pressure, the volume is diminished by Q, and the pressure becomes therefore $PV/(V-Q)$, or, if Q be small, $P(1+Q/V)$. The work done in compressing the air, being equal to the product of the average pressure and the diminution of volume, is therefore $(1+Q/2V)PQ$, and the total increase of potential energy, so far as it depends

on the square of Q, is $\frac{1}{2}g\rho Q^2(a^{-1}+hV^{-1})$, where P, the original air pressure, has been expressed in terms of h, the height of its equivalent column of mercury.

We shall find throughout this investigation that the air pressure which may exist in the various tubes above the mercury surface may be taken into account simply by replacing a (the cross-section of the tube at the surface of the mercury) by a_1, where

$$a_1^{-1} = a^{-1}+hV^{-1}. \tag{1}$$

We find finally for the total potential energy

$$g\rho h_0 Q+\tfrac{1}{2}g\rho Q^2/a_1.$$

It follows from the laws which regulate the flow of viscous fluids through tubes that in order to maintain a constant flow dQ/dt, the difference in the pressure at the ends of a tube of length l and cross-section s, is $8\pi\eta ls^{-2}(dQ/dt)$, where η is the coefficient of viscosity. The rate at which energy is being dissipated in the series of tubes which are joined together is therefore $8\pi\eta(dQ/dt)^2\Sigma ls^{-2}$. We form the equation of motion by expressing that the sum of (1) the rate of increase in potential energy, (2) the rate of increase of potential energy, and (3) the rate at which energy is dissipated, must be equal to the work from outside, and this leads to

$$\frac{4}{3}\rho\Sigma(ls^{-1})\frac{d^2Q}{dt^2}+8\pi\eta\Sigma(ls^{-2})\frac{dQ}{dt}+\frac{g}{a_1}\rho Q = p_0-g\rho h_0. \tag{2}$$

Here p_0 denotes the pressure on the free outer surface AB.

In general we may write $p_0 = g\rho h_0 + p$, where p is the variable part of the pressure. When p is zero the right-hand side vanishes, and we obtain the equation for the free oscillation of the tube.

For a number of tubes connected either directly or through a reservoir (fig. 2) we can form the equations appropriate to each of the tubes separately. If the velocities with which the mercury is transferred in the reservoir from one tube to another be small compared with the velocities on which the kinetic energies in the tube depend, we may take the pressure in the reservoir at the entrance of the tubes to be the same for all tubes. In order not to complicate the investigation, we shall assume that the number of tubes is three, and that the kinetic energy of two of them depends only on the kinetic energy of the capillary portions. For convenience of reference tubes 1, 2, and 3 will be called "contact tube," "barometer tube," and "index tube" respectively.

The equation regulating the motion in the first tube then reduces to

$$\frac{d^2Q_1}{dt^2}+\lambda_1\frac{dQ_1}{dt}+n_1^2Q_1 = 3ps_1/4l_1\rho, \tag{3}$$

where $\lambda_1 = 6\pi\eta/\rho s_1 = 0{\cdot}0222/s, \qquad n_1^2 = 3gs/4a_1l_1,$

with similar equations for the other two tubes. If we assume Q_1, Q_2 and Q_3 to vary as $e^{i\kappa t}$, the equations become

$$(n_1^2 - \kappa^2 + i\lambda_1\kappa)Q_1 = a_1n_1^2p/g\rho,$$
$$(n_2^2 - \kappa^2 + i\lambda_2\kappa)Q_2 = a_2n_2^2p/g\rho,$$
$$(n_3^2 - \kappa^2 - i\lambda_3\kappa)Q_3 = a_3n_3^2p/g\rho,$$

subject to the condition

$$Q_1 + Q_2 + Q_3 = 0.$$

We derive from this

$$\frac{a_1n_1^2}{n_1^2 - \kappa^2 + i\lambda_1\kappa} + \frac{a_2n_2^2}{n_2^2 - \kappa^2 + i\lambda_2\kappa} + \frac{a_3n_3^2}{n_3^2 - \kappa^2 + i\lambda_3\kappa} = 0. \tag{4}$$

. By means of this equation, the motion of the mercury in the three tubes, if disturbed from its position of equilibrium and left to itself, may be determined. At present we may content ourselves with noticing an interesting analogy to optics. If we neglect viscosity, the two possible frequencies of free oscillation are related to the free periods of the three tubes separately as the wave velocities in biaxal crystals to the three principal velocities, if the direction cosines of the wave normals in the latter case are made to be proportional to the radii of the sections of the tubes at the free surfaces where the areas are a_1, a_2, a_3 respectively.

3. Before passing on to our main problem, we may introduce a condition which results in a considerable simplification of the mathematical treatment. Referring to fig. 2 in Dr. Duffield's paper, it is seen that, as the mercury flows down the index tube, it has to overcome the capillary resistance in the two other tubes. It is essential that, at the moment contact is made at C, the level at H should be that applying to a state of equilibrium according to the laws of hydrostatics. Hence, either the mercury must be introduced so slowly that hydrostatic equilibrium has time to establish itself, or the length of capillaries in the barometer and contact tubes must bear such a relation to each other that the difference in the level of mercury in both tubes remains at its hydrostatic value as the mercury rises. It is easy to fulfil this last condition. If the pressure in the air reservoir were to remain constant, the quantities which would have to enter the barometer and contact tubes would have to be in proportion to the areas of the cross-sections, and this relation still holds true when the compression of the air is taken into account, by introducing the reduced areas of cross-sections as deduced in §1, so that we may put $Q_1/a_1 = Q_2/a_2$. It is understood that, unless otherwise stated, a_1 always refers to this reduced cross-section.

The barometer tube is supposed to have a vacuum above it, so that a_2 is the actual cross-section.

Referring to equation (3), and the corresponding one applying to the barometer tube, we may eliminate p, and obtain

$$\left(\frac{l_1 d^2 Q_1}{s_1 dt^2} - \frac{l_2 d^2 Q_2}{s_2 dt^2}\right) + \left(\frac{l_1 dQ_1}{s_1^2 dt} - \frac{l_2 dQ_2}{s_2^2 dt}\right) 6\pi\eta + \tfrac{3}{4} g \left(\frac{Q_1}{a_1} - \frac{Q_2}{a_2}\right) = 0. \qquad (5)$$

The condition of hydrostatic equilibrium between the two tubes being $a_2 Q_1 = a_1 Q_2$, the equation shows that, if initially the velocities be zero so that this condition is established, it will be maintained throughout the motion, provided that $s_1 = s_2$, and $l_1 a_1 = l_2 a_2$. In Dr. Duffield's apparatus, $s_1 = s_2$, but the length of the capillary in tube 1 is too small to satisfy the second condition: it will be important to remedy this in future designs.

Under the conditions of the experiment, the term depending on the second differential coefficient of (5) is of comparatively small importance. Neglecting the effects of inertia, we may consequently drop the condition $s_1 = s_2$, and replace the second condition by $l_1 a_1 s_1^{-2} = l_2 a_2 s_2^{-2}$. But it is to be remembered that this fulfils the required condition only approximately.

4. If the ship in which the apparatus is placed be subject to a vertical acceleration, this will be equivalent to a change of gravity, and it is necessary to discuss how far the accuracy of the experiment may be interfered with. We are here concerned only with periodic changes, and the forced oscillations of the mercury which result from these changes. In equation (1) we must add a periodic term, say $-\gamma \cos \omega t$, to g, but taking account only of small quantities of the first order, we shall neglect the product γQ. We may treat therefore the effect of γ to be that of an impressed force $\gamma \rho h_0 \cos \omega t$, and for h_0 we must substitute h_1, h_2, h_3, in the three tubes respectively. These quantities denote the heights of the columns of mercury in the tubes measured from a fixed level, and it is obvious that only their differences can enter into the results.

The resulting equations are

$$\left.\begin{aligned} \frac{l_1}{s_1}\left(\frac{d^2 Q_1}{dt^2} + \lambda_1 \frac{dQ_1}{dt} + n_1^2 Q_1\right) - \frac{3p}{4\rho} &= \tfrac{3}{4}\gamma h_1 \cos \omega t, \\ \frac{l_1}{s_2}\left(\frac{d^2 Q_2}{dt^2} + \lambda_2 \frac{dQ_2}{dt} + n_2^2 Q_2\right) - \frac{3p}{4\rho} &= \tfrac{3}{4}\gamma h_2 \cos \omega t, \\ \frac{l_3}{s_3}\left(\frac{d^2 Q_3}{dt_2} + \lambda_3 \frac{dQ_3}{dt} + n_3^2 Q_3\right) - \frac{3p}{4\rho} &= \tfrac{3}{4}\gamma h_3 \cos \omega t, \end{aligned}\right\} \qquad (6)$$

if, as is assumed, the capillaries in the barometer and contact tubes have the

same bore $\lambda_1 = \lambda_2$. We now write $Q_1 e^{i(\omega t+\alpha_1)}$ for Q_1; $Ke^{i(\omega t+\beta_1)}$ for $3p/4\rho$, and further

$$\begin{array}{lll} Q_1 \cos\alpha_1 = x_1, & Q_2 \cos\alpha_2 = x_2, & Q_3 \cos\alpha_3 = x_3, \\ Q_1 \sin\alpha_1 = y_1, & Q_2 \sin\alpha_2 = y_2, & Q_3 \sin\alpha_3 = y_3, \\ c_1 = l_1(n_1^2-\omega^2)/s_1, & c_2 = l_2(n_2^2-\omega^2)/s_2, & c_3 = l_3(n_3^2-\omega^2)/a_3, \\ f_1 = \lambda_1 l_1\omega/s_1, & f_2 = \lambda_2 l_2\omega/s_2, & f_3 = \lambda_3 l_3\omega/a_3, \end{array}$$

$$K\cos\beta = u_1, \qquad K\sin\beta = u_2,$$

$$E_1 = \tfrac{3}{4}\gamma h_1, \qquad E_2 = \tfrac{3}{4}\gamma h_2, \qquad E_3 = \tfrac{3}{4}\gamma h_3.$$

For the eight unknown quantities x, y, and u, we have now the following six equations resulting from separating the real and imaginary parts of (6), and two resulting from the condition $Q_1+Q_2+Q_3 = 0$:—

$$\left.\begin{array}{ll} c_1x_1-f_1y_1+u_1 = E_1, & f_1x_1+c_1y_1+u_2 = 0 \\ c_2x_2-f_2y_2+u_1 = E_2, & f_2x_2+c_2y_2+u_2 = 0 \\ c_3x_3-f_3y_3+u_1 = E_3, & f_3x_3+c_3y_3+u_2 = 0 \\ x_1+x_2+x_3 = 0, & y_1+y_2+y_3 = 0 \end{array}\right\}. \tag{7}$$

The solution is best expressed in terms of the auxiliary quantities

$$p = c_2f_1-c_1f_2, \qquad q = c_3f_2-c_2f_3, \qquad r = c_1f_3-c_3f_1.$$

The solution has the form

$$\begin{aligned} Sx_1 &= \Delta_1^2(E_2-E_1)+\Delta_1^3(E_3-E_1), \\ Sx_2 &= \Delta_1^2(E_1-E_2)+\Delta_2^3(E_3-E_2), \\ Sx_3 &= \Delta_1^3(E_1-E_3)+\Delta_2^3(E_3-E_3), \end{aligned}$$

with similar equations for y_1, y_2, y_3, in which the three quantities Δ_1^2, etc., are replaced by similar ones D_1^2, etc.

For these quantities we find

$$\begin{array}{ll} \Delta_1^2 = (f_1+f_3)q-(f_2+f_3)r-c_3(C+F), & D_1^2 = (c_1+c_3)q-(c_2+c_3)r+f_3(C+F), \\ \Delta_1^3 = (f_3+f_2)p-(f_1+f_2)q-c_2(C+F), & D_1^3 = (c_2+c_3)p-(c_1+c_2)q+f_2(C+F), \\ \Delta_2^3 = (f_2+f_1)r-(f_3+f_1)p-c_1(C+F), & D_2^3 = (c_1+c_2)r-(c_1+c_3)p+f_1(C+F), \end{array} \tag{8}$$

where $\qquad C = c_1c_2+c_2c_3+c_3c_1, \qquad F = f_1f_2+f_2f_3+f_3f_1;$

also $\qquad S = (C+F)^2+p\,(p-q-r)+q\,(q-r-p)+r\,(r-p-q).$

The numerical calculation is simplified by the relation which must hold to satisfy the conditions discussed in § 3. If $s_1 = s_2$, and accordingly $a_1l_1 = a_2l_2$, it follows that $n_1^2 = n_2^2$, and $p = 0$. When this condition is fulfilled, the oscillation in the index tube can be reduced to zero by a proper adjustment of

the length of the columns of mercury. From the definition of p, q, and r, it follows that

$$c_3p+c_1q+c_2r = 0,$$

and if $p = 0$, $c_1q = -c_2p$. Hence in that case $\Delta_1^3 : \Delta_2^3 = c_2 : c_1$, $D_1^3 : D_2^3 = f_2 : f_1$. As $p = 0$ involves that $c_2 : c_1 = f_2 : f_1$, we can reduce both x_3 and y_3 to zero, if

$$\frac{E_2-E_3}{E_3-E_1} = \frac{h_2-h_3}{h_3-h_1} = \frac{c_2}{c_1} = \frac{f_2}{f_1}.$$

There would be some convenience, if the oscillation due to the vertical motion were made as small as possible, although it cannot be destroyed altogether, because during the experiment the level of the mercury in the index tube varies, and the amplitude is only zero for one definite position. When the mercury occupies that position, the amplitudes of the oscillations of the other tubes may be obtained in a very simple way. For, from equations (7), it follows that if x_3 and y_3 are zero, $u_1 = E_3$. This gives:—

$$c_1x_1-f_1y_1 = E_1-E_3,$$
$$f_1x_1+c_1y_1 = 0,$$
$$(c_1^2+f_1^2)x_1 = c_1(E_1-E_3),$$
$$(c_1^2+f_1^2)y_1 = f_1(E_1-E_3),$$

and the amplitude of oscillation of Q, which is $(x^2+y^2)^{\frac{1}{2}}$, becomes

$$\left.\begin{aligned} &(E_3-E_1)(c_1^2+f_1^2)^{-\frac{1}{2}}, \\ &= (E_2-E_1)(c_1^2+f_1^2)^{-\frac{1}{2}}f_1/(f_1+f_2), \\ &= (E_2-E_1)(c_1^2+f_1^2)^{-\frac{1}{2}}a_2/(a_1+a_2) \\ &= (E_2-E_1)(c_2^2+f_2^2)^{-\frac{1}{2}}a_1/(a_1+a_2) \end{aligned}\right\}. \qquad (9)$$

Q here stands for the amplitude of "volume." To obtain the corresponding amplitude of level, we must divide by the respective cross-sections, which, however, now are the actual and not the "reduced" ones.

5. We are now in a position to discuss the effects of experimental errors, so far as they are affected by the relative dimensions of the various parts of the apparatus. Much depends on the regularity of the electrical contact, and though Dr. Duffield's experiments on this point seem reassuring, we must be prepared for small variations. If the contact be slightly delayed owing to a speck of dust or otherwise, so that the level in the contact tube is too high by a quantity δh_1, when the signal is given, the level in the barometer tube will be too high by the same amount, so that the total quantity of mercury in the two tubes will exceed the proper amount by $(a_1+a_2)\,\delta h_1$, with the result that the level in the index tube will be the same as if, the contact being made at the right time, the mercury in the barometer tube stood by an

amount $(a_1+a_2)\,\delta h_1/a_2$ above its correct height. The error in the measured value of $\delta g/g$ is therefore $(a_1+a_2)\,\delta h_1/a_2\,(h_1-h_2)$. In Dr. Duffield's experiment a_1/a_2 is 3·88, so that the proportional error in the value of gravity would be nearly $5\delta h_1\,(h_1-h_2)$. If his estimate that δh is not greater than 2×10^{-4} cm. be correct, the maximum error of $\delta g/g$ would be 5×10^{-5}, for $h_1-h_2=20$.

It is desirable to aim at an accuracy of about 2×10^{-5}, and therefore to reduce the area of the mercury surface in the contact tube. In doing so, however, the relation $a_1l_1=a_2l_2$ must, if possible, be maintained. The difficulty lies in lengthening l_1 without unduly increasing the length of the whole apparatus, and introducing a large quantity of dead mercury. To overcome this difficulty, two ways are open to us. The capillary in the first tube, instead of being vertical, might be horizontal or of spiral form; this has the disadvantage of diminishing the rigidity of the apparatus. The alternative is to be satisfied with an approximate fulfilment of the condition discussed in § 3, and to make the capillary in the contact tube of smaller bore, so that $a_1l_1s_1^{-2}=a_2l_2s_2^{-2}$. If $l_2=4l_1$, and $s_2=2s_1$, the areas of a_1 and a_2 would now be equal, and the error due to a delayed contact would be reduced in the ratio 5 : 2. This manner of overcoming the difficulty should, however, be avoided if possible.

As regards the error due to the vertical motion of the ship, the oscillation of the mercury may cause the contact to be accelerated or retarded. If the mercury at the moment of contact stand at a height δh above its normal height, and the adjustment be such that there is no oscillation in the index tube, the mercury in the barometer tube stands too low by an amount $a_1\delta h/a_2$. The error in the difference of level (h_2-h_1) is $(a_1+a_2)\,\delta h/a_2$. If δh is at its maximum, and therefore equal to the amplitude of oscillation, we may substitute its value from (9) and the error in $\delta g/g$ is $(c_1^2+f_1^2)^{-\frac{1}{2}}\times(E_2-E_1)/(h_2-h_1)$ or $\frac{3}{4}\,\gamma/a(c_1^2+f_1^2)^{\frac{1}{2}}$. In the actual cases f_1 is always large compared to c_1, and proportional to l_1, so that what is required is to make a_1l_1, and hence also a_2l_2, as large as possible.

Dr. Duffield has shown that the volume of mercury in the reservoir necessary to secure compensation for variations of temperature is proportional to a_2, and it is desirable to keep this as low as possible. As constructional considerations forbid an undue lengthening of the apparatus, the effects of the vertical oscillation must be made to depend mainly on the narrowness of the capillary bore. In several respects an advantage would be secured by transferring the contact arrangement to the barometer tube, but this would preclude the possibility of a temperature compensation, and therefore need not be further considered.

We may sum up our conclusions as follows:—The effects of "irregular

contacts" are diminished by reducing the cross-section of the contact tube as compared with that of the barometer tube; the effect of the vertical oscillation of the ship is diminished by increasing both cross-sections; but an upper limit to these areas is set by the necessity of reducing the quantity of mercury required for the temperature compensation, on account of the difficulty of securing uniformity of temperature throughout its mass. A simple calculation shows that the average temperature of the mercury reservoir should not differ by more than the hundredth of a degree from the temperature of the air reservoir.

If the shape of the mercury reservoir be modified suitably so as to increase its surface, the diameter of about 4 mm., adopted in Dr. Duffield's barometer tube, might perhaps be increased, but not to more than 6 mm. The area of the contact tube should also be as small as constructional considerations allow us to make it. If we wish to maintain rigidly the condition that the flow of mercury in the barometer and contact tubes be such that the difference in level is always that of hydrostatic equilibrium, the cross-sections of the capillaries should be equal, and the lengths of the capillaries should be inversely as the reduced cross-sections of the tubes at the free surfaces of the mercury. As the difference in level between the point of contact and the surface of the mercury in the barometer tube is fixed at about 20 cm., we cannot easily make the length l_1 greater than one-third that of l_2. In that case a_2 would be three times a_1, and an error of contact would cause an error twice as great as if a_1 and a_2 were equal to each other. If this difference be considered to be important, we have no choice left but to give up partially the theoretically correct relative flow in the two tubes, and make the capillary in the contact tube smaller than that in the barometer tube. The matter then turns on the importance of the relationship discussed in § 3. In Dr. Duffield's apparatus this consideration is neglected, and according to him there has been no appreciable loss of accuracy in this respect, because the flow of mercury was so slow that hydrostatic equilibrium was always sensibly established. It is to this point I wish to draw attention. In order to eliminate errors of observation, and notably those due to the vertical oscillation of the ship, it is highly desirable that each measurement should not occupy much time, so that it can be repeated at short intervals. It is here that the advantage of the adjustment in the resistance to the flow in the two tubes will show itself.

In Dr. Duffield's apparatus the cross-section of the capillary is $6{\cdot}4 \times 10^{-4}$ cm. Calculation shows that it may safely be reduced to about a third of that value if the relations $a_1 l_1 = a_2 l_2$, and $s_1 = s_2$ be maintained. It follows that $n_1 = n_2$, and equation (3) leads to a quadratic giving two values for κ, so

that the movement of the mercury in each of the three tubes may be expressed in the form

$$x = \mathrm{A}e^{-\kappa_1 t} + \mathrm{B}e^{-\kappa_2 t},$$

where x is the height of the column above its equilibrium position. The values of κ depend on the viscosity of mercury, the ratio a_1/a_2, and the quantities s, a_3, l_2, and l_3. Taking l_2, which is the length of its capillary, as 20, and its cross-section as 2×10^{-4}; also $s/a_3 = 0{\cdot}1$, which gives for a_3 the value it has in Dr. Duffield's apparatus; further, $a_1 = 3a_2$:

I find for $l_3 = 10$: $\kappa_1 = 94$, $\kappa_2 = 0{\cdot}113$,

and for $l_3 = 20$: $\kappa_1 = 82$, $\kappa_2 = 0{\cdot}115$.

The term involving κ_1 has very little influence on the motion, even in the initial stages, and taking the fraction $(\kappa_1 - \kappa_2)/\kappa_1$ equal to unity, the position of the mercury in the index tube is represented sufficiently accurately by $x = x_0 e^{-\kappa_2 t}$. If the pressure be adjusted so that the mercury in the index tube stands, *e.g.*, 1 cm. above the position at which the electric connection is established, and is then increased so that the position of equilibrium stands 1 cm. below the same position, we must put $x_0 = 2$, and contact will take place when $x = 1$. This gives a value of t equal to 6 sec. To repeat the experiment, the pressure above the index tube must again be diminished, and some little time given for the mercury to come to rest; but one measurement per minute could easily be carried out in this manner.

6. The error in $\delta g/g$ may be calculated in special cases from the formulæ which have been given. I have constructed some tables for this purpose, but it is not necessary to reproduce them here, because the figures depend very much on the detailed dimensions of the apparatus. The order of magnitude is, however, easily obtained from the simplified equations which hold when the oscillation of the index tube is zero. In that case the error has been found to be $3\gamma/4a_1(c_1^2+f_1^2)^{\frac{1}{2}}$. Here γ stands for the vertical acceleration of the ship, and if the vertical displacement is represented by $\mathrm{R}\cos\omega t$, we may write $\omega^2\mathrm{R}$ for γ. The value of c_1 is of the order of one-hundredth that of f_1, so that we may neglect it, and by substituting the value of f_1 we obtain for the error in $\delta g/g$ the expression $3\omega\mathrm{R}s/4\lambda_1 l_1 a_1$, or $3{\cdot}4 \times \omega\mathrm{R}s^2/l_1 a_1$, if λ_1 be substituted from (3). If we adopt 1·5 as the maximum allowable area for a_1, and make the ratio a_1/a_2 equal to three, we find $l_1 a_1 = l_2 a_2 = 10$. (Note that a_1 is the "reduced" area determined by (1).) With a cross-section of 2×10^{-4} for the capillary, the error in gravity takes the value $1{\cdot}36 \times 10^{-7}\mathrm{R}\omega$.

Hecker finds the period of the waves responsible for the vertical motion of the ship to be about six seconds, so that ω may be replaced by unity. A

vertical oscillation of the ship having an amplitude of a metre would therefore introduce an error in $\delta g/g$ of less than $1{\cdot}4 \times 10^{-5}$ in the most unfavourable case, where contact is made when the oscillation of the mercury is at its maximum. The error may, no doubt, be reduced by taking the mean of a number of observations, but we must remember that the velocities of flow are smallest when the displacement is near its maximum value. In a single observation there is an equal chance that the displacement, and consequently the error, is above or below $(\cos 45^\circ)^{-1}$, or $0{\cdot}7$ of the maximum.

A simple calculation further shows that when the number (n) of observations is large, their mean value has a probable error $0{\cdot}477 n^{-\frac{1}{2}} x$, where x stands for the maximum error of a single observation. For $n = 9$ this is equal to $0{\cdot}16 x$, or about the sixth part of the maximum. Even in that case there would still be a probability of 1 in 200 that the error exceeds two-thirds of the maximum. Some allowance must also be made for irregularities in the motion of the ship, which will never be simply periodic. It may be added that the dimensions of Dr. Duffield's apparatus give, for the error due to the motion of the ship, values about 20 times larger than those in the above example, on account of the larger cross-section of the capillaries and the smaller cross-section of the upper part of the barometer tube.

7. Insufficient attention has, I think, been given by Dr. Duffield to the angular displacements of the barometer system, which, owing to the ship's motion, cannot be trusted to remain perfectly vertical. The matter is important, because the error cannot be eliminated by repeating the observations. Whether the deviation be to the right or left, the mercury will always rise in the barometer tube, and for a deviation α the apparent height of the barometer will be $h/\cos \alpha$, when its real height, *i.e.* the vertical difference of level in the barometer and contact tube, is h. A deviation of 1° would consequently cause an error of $1{\cdot}5 \times 10^{-4}$ in the value of $\delta g/g$. To reduce the error to 2×10^{-5}, the barometer system should never deviate more than 22 minutes of arc from the vertical. Substantial improvements could, I think, be made in the method of suspension, and it may be advisable to adopt some such device as that used by Mr. Hecker for measuring the deviations from the vertical.

8. In conclusion, a few suggestions may be helpful in designing an apparatus for future use. The most important points are to secure the temperature correction and to preserve at the same time the relation

$$l_1 = a_2 l_2 \left(\frac{1}{a_1} + \frac{h}{V} \right) \tag{10}$$

where a_1 now stands for the actual cross-section of the contact tube. When

the volume of the air vessel is given, l_1 fixed at, say, 20 cm., and the barometer tube selected so that a_2 is also fixed, h may provisionally be taken to be 20, and the area, a_1, of the contact tube calculated by assuming $l_2 = \frac{1}{3}l_1$. The tubing for the contact tube may then be selected so as to come near to this value. The accurate value of a_1 is then found by measurement. The length, h, may next be fixed more accurately in the following manner. The volume of the mercury reservoir necessary for the temperature correction having been calculated, and constructed as accurately as possible, the outstanding error is measured. If necessary, the formula $\alpha v = a_2h/\mathrm{T}$, where T is the temperature of observation, α the coefficient of expansion of mercury relative to glass, and v the total volume of mercury, may now be applied to obtain a correction for h, and substituting this corrected value of h into (10) we may determine the value of l_1. In calculating the volume v, no account should be taken of the mercury which stands in the barometer tube above the level of the point of contact, because its dilatation with temperature is automatically compensated by its diminished density. The value of h determines the pressure in the air reservoir.

It will also be advisable to provide for a sufficient volume for the vacuum at the upper end of the barometer tube, so that small quantities of air which may lift or gradually accumulate in it should not substantially alter the measurements.

Finally, the pressure in the tubes E and F (fig. 2 of Dr. Duffield's paper) should be adjustable independently, so that when the taps E and F are opened, the rate of flow can be settled beforehand. This is especially important with regard to E, so as to secure a proper control over the rate of flow in the index tube. The tap F would be used only when, owing to variations in the value of gravity, a measured quantity of mercury has to be either added to or subtracted from the apparatus.